Chance Encounters - 20 years later
Fundamental ideas in teaching probability at school level

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Summary

This paper considers how probability is now taught in England and the way that the curriculum reflects key research ideas from the last three decades. Links are made to work undertaken in probability education and the way that challenges in the book, Chance Encounters, have been met. This is based on the current curriculum and also the performance of children in tests. The conclusion notes that there is some way to go in ensuring children are well versed in probability.

Introduction

It is two decades since the publication of Chance Encounters (Kapadia and Borovcnik, 1991); the title was chosen as slightly whimsical but the underlying content of the teaching of probability was promoted and assessed critically and carefully. This was clear from the sub-title of Probability in Education. The book was written to collect and share findings from major researchers across Europe and North America on key aspects to the teaching of probability internationally. It grew out of collaborative work at the first two International Conferences on the Teaching of Statistics (ICOTS) in the 1980s, in Sheffield and Vancouver respectively, supported by smaller meetings elsewhere. A deliberate attempt was to take perspectives from various viewpoints, not least from the origins of the subject with regards to games, gambling, and counter-intuitive paradoxes. Links were made to the current curricula at the time, including the National Curriculum of Teachers of Mathematics (NCTM) in the United States. Attention was paid to key psychological research and to the growing influence of computers.

In that book a number of challenges were set. This presentation aims to review those challenges, with specific commentary on the situation in England. This is not intended as an exhaustive study but rather to make some comments to provoke and stimulate debate. Many papers, books and articles have been published but no attempt is made at a summary or review. The focus in England has changed towards a greater emphasis on statistics and statistical methods, particularly the representation of data, partly at the expense of probability. Probability is taught in a relatively limited way and not systematically linked as the underpinning theory for statistical distributions.

Yet risk and management of risk is being seen as increasingly important. The research of Tversky and Kahnemann (1974) is recognised as ground-breaking, not least with the award of the Nobel prize. Its importance and relevance in economics is growing. Yet education has not fully incorporated the underlying ideas.

There were ten provocative statements relating to how people view, use and process ideas relating to probability. What has changed? What is actually taught and what can children do currently? The responses to these questions will relate to current policy in England. Since this is a panel discussion, the aim is to stimulate responses from those in other countries about their own perspectives, either in debate or by subsequent written communication. This paper gives some thoughts but, more importantly, it aims to promote a more sustained debate.

Discussion

Below (in Appendix 1) are the attainment targets and related programmes of study in England for probability. These are for pupils at the end of primary school / Year 6 or age 11 (Level 4) and subsequently: Level 5/6 is for Year 9 or age 14, while Level 6/7 is for Year 11 or age 16. Exceptional performance is for pupils with an ability range up to the top 15th percentile. There is also information (in Appendix 2) about performance in questions on probability in tests undertaken at age 14. The attainment targets describe what pupils are expected to learn and occur in tests. The more detailed programmes of study include more general information on what should be studied.

A key point to note in terms of children up to Year 6 (working at Level 4) is that there is no reference in the attainment targets to probability, even at a basic level. This is a marked change from the curriculum as originally devised twenty years ago, when even infants in Year 2 were expected to investigate some early notions of chance. At that time the assessment criteria were very detailed and specific, with a whole set of targets which related solely to probability; at the time it was attainment target 14, whilst attainment targets 12 and 13 related to statistics. Now there is a set of targets devoted to handling data which is seen to encompass probability. In some ways the merger could be seen in a positive light, except that some key ideas of probability (such as subjective notions) were removed. Most importantly, because of external pressures on an already crowded curriculum, references to probability were initially removed from the content for infants and then for all children up to Year 6 (age 11). These changes were implemented early this century and so it would be interesting to conduct tests on children at different stages of school to track any effects.

Probability has been retained for secondary age pupils as can be seen in the content for Level 5. Work would begin in Year 7 and continue for the next year on the ideas of probability derived from experimental situations only. While this is a standard way of addressing probabilistic notions, it is unfortunate that no direct mention is made of subjective notions, particularly in situations where an experiment is not possible. Moreover, it is also important to confront children's early conceptions and misconceptions of probability. This would normally be taught in schools (as illustrated in the programmes of study), but explicit mention of subjective probability would have been particularly helpful.

The next level (Level 6 for Year 9) for probability develops into combined events and notions of mutual exclusiveness. It is only at Level 7 for Year 11 that mention is made of distributions and inference, though links with probability are not explicitly mentioned. It is only for the able pupils that ideas of independence are explicitly studied, as well as aspects of sampling, with links to reliability. In practice, sampling would be introduced earlier for surveys, but the key links between probability and statistics may not be made explicitly.

Perhaps the most important omission is any reference to risk and hence to real-life applications. It is certainly true that the majority of teachers would make links to real-life situations, indeed they often occur in text-books. But the lack of reference to risk is very important in the context of the current world.

It is for this reason that the research of Tversky and Kahnemann has come to such prominence. It is recognised as a key skill in many situations; too often it is a key skill which is lacking, even in students who study for higher education and will therefore go on to the key posts of influence and responsibility.

It should be noted that there are no specific attainment targets in probability for Year 6 pupils, though ideas of equal likelihood are included in the programmes of study described below. Since the tests for Year 6 pupils are perceived as 'high-stakes' tests which are used to gauge the performance of a school, it may be that younger children do not now get the exposure to underlying ideas of probability sufficiently.

It is also crucial to note that there is virtually no reference to subjective probability at any stage. This omission is very significant for various reasons. Subjective notions of probability are the first notions that many children develop, albeit on an informal basis. Indeed we all use informal probabilistic notions daily in making certain decisions. Thus it is important when studying ideas of probability in the classroom, some account should be taken of such perceptions. This is a fundamental way of tackling misconceptions. One common and immediate such misconception relates to the chance of getting a six when throwing a die. This is linked to waiting times for a desirable event; the waiting seems to increase the internal perception of likely occurrences. In all research in mathematical education, it is recognised that early conceptions of ideas need to be discussed so that children can develop better understanding. Probability is no different and early notions as well as misconceptions need to be addressed. This takes time which is not fully recognised.

The approach taken in these attainment targets and programmes of study is still quite firmly based on relatively formal and mathematical notions of probability, where contexts and underlying subjective knowledge is to be ignored. In some ways, this is linked to the fact that it is easier to set questions in such contexts. The questions are seen as reliable and consistent in testing concepts and ideas accurately. However, it is a moot point as to how well children can use such ideas in the future for real-life situations. This becomes clear when looking at the performance on the test questions and children's responses given below.

In the Appendix 2 below are quotations from analysis of test results relating to probability of Year 9 pupils in 2001; as noted above, they would have encountered ideas of probability in their earlier primary education. It will be interesting to compare the performance on similar ideas of those who took the tests next year, as they would not have encountered ideas of probability in earlier primary school education.

In particular the analysis of facility on questions does show good intuitive understanding of the basic underling notions of probability, particularly in experimental situations. The difficulties seem to arise in relation to pupils' facility and knowledge of other ideas in arithmetic and geometry. There is also evidence of the use of doubling rather than squaring even when it may lead to wrong answers such as a probability exceeding unity.

For the first set of questions, equal likelihood in simple situations is effectively used, with the correct notation by the vast majority (around three quarters) who reach Level 5 in Year 9. The low facility for the third question, Coloured Cubes, may well be related to poor facility with fractions rather than notions of probability.

The more discriminating and interesting test for equal likelihood occurs in 2 (Spinners). However, the errors children make are linked to using the notion of area rather than angle in the question. In fact area is often the right measure to use in determining equal likelihood. In this particular case, one spinner is made bigger than the other one, almost deliberately to trick children. Whilst there are times when a trick is sensible to use to search for underlying understanding, this is not apparent here. The key concept is to either choose one of five or one of six equally likely outcomes. For such a situation, to look for underlying probabilistic understanding, it would be better to have spinners of similar size; this would make the item less discriminating with regards to children's facility with the question, but more closely linked to the underlying ideas. For this question, it is also interesting that children are asked to give a reason to explain their answer. This is particularly powerful to help to understand a child's reasoning and used too rarely. In this case, the notion of the starting point of the arrow is a misconception which needs to be tackled from a probabilistic perspective, particularly if it accounts for a significant proportion of children.

The question on Pots is a typical one about combining probabilities, perhaps by using a tree diagram. This question is aimed at the high ability pupils, typically the top half of the population. The first part is an interesting and sensible test of deciding when two probabilities should be added or multiplied in a given situation. It is a key idea in developing on from the probability of a single event using equal likelihood or frequentist notions of probability. Since two fifths of pupils incorrectly added, this implies that almost three quarters (a very high proportion) of Year 9 children have incorrect notions of the fairly fundamental notions of when it is inappropriate to add two probabilities. The final part of the question is certainly quite difficult and challenging. The facility for the full answer is very low and represents about the top percentile of the population, whilst around the top 5% got the question partially right in forgetting that either the first or the second pot could crack.

With regards to the challenges set out in Chance Encounters, the curriculum as set out in its assessment objectives, does not show signs of addressing key areas of misconceptions, which was a vital strand running through Chance Encounters. This indicates that insufficient notice has been taken of the psychological issues explored, or the teaching sequences discussed. Another notable omission relates to computers, though such references would, more properly, come in the supporting curriculum materials.

With regards to the ten provocative statements at the end of the book, most remain all too true. Elements of being haphazard and incomplete in dealing with probability are clear and people use memorable events to process information. Very low and high probabilities are hard to assess. Frequentist ideas underlie much of the approach to probability and people are getting better in using such information.

Conclusion

In England, at least, there were positive steps taken towards the teaching of probability twenty years ago, starting with young children on the intuitive underlying ideas. This has changed and is now mainly for secondary age pupils, where a more formal approach is taken. There is insufficient attention given to the fundamental ideas in probability and in addressing misconceptions. The test results show that pupils do gain facility in situations where equal likelihood is assumed. The errors made indicate that subsequent ideas of combining probabilities are challenging to most pupils. The errors also show that misconceptions on key ideas remain.

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Appendix 1: National Curriculum: Attainment Targets 2001 – Mathematics Key Stage 3

www.qca.org.uk/curriculum

Handling Data and Probability

Level 4 (Year 6, grade 5, age 11)

Pupils collect discrete data and record them using a frequency table. They understand and use the mode and range to describe sets of data. They group data in equal class intervals where appropriate, represent collected data in frequency diagrams and interpret such diagrams. They construct and interpret simple line graphs.

Level 5 (Year 9, lower ability, grade 8, age 14)

Pupils understand and use the mean of discrete data. They compare two simple distributions using the range and one of the mode, median or mean. They interpret graphs and diagrams, including pie charts, and draw conclusions. They understand and use the probability scale from 0 to 1. They find and justify probabilities and approximations to these by selecting and using methods based on equally likely outcomes and experimental evidence, as appropriate. They understand that different outcomes may result from repeating an experiment.

Level 6 (Year 9, average ability, grade 8, age 14)

Pupils collect and record continuous data, choosing appropriate equal class intervals over a sensible range to create frequency tables. They construct and interpret frequency diagrams. They construct pie charts. They draw conclusions from scatter diagrams, and have a basic understanding of correlation. When dealing with a combination of two experiments, they identify all the outcomes. When solving problems, they use their knowledge that the total probability of all the mutually exclusive outcomes of an experiment is 1.

Level 7 (Year 9, high ability, grade 8, age 14)

Pupils specify hypotheses and test them by designing and using appropriate methods that take account of variability or bias. They determine the modal class and estimate the mean, median and range of sets of grouped data, selecting the statistic most appropriate to their line of enquiry. They use measures of average and range, with associated frequency polygons, as appropriate, to compare distributions and make inferences. They understand relative frequency as an estimate of probability and use this to compare outcomes of experiments.

Level 8 (Exceptional performance, Year 11, grade 10, high ability)

Pupils interpret and construct histograms. They understand how different methods of sampling and different sample sizes may affect the reliability of conclusions drawn. They select and justify a sample and method to investigate a population. They recognise when and how to work with probabilities associated with independent, mutually exclusive events.

Numbers and the Number System

Level 6 (Year 9, average ability, grade 8, age 14)

Pupils order and approximate decimals when solving numerical problems and equations, using trial and improvement methods. They evaluate one number as a fraction or percentage of another. They understand and use the equivalences between fractions, decimals and percentages, and calculate using ratios in appropriate situations. They add and subtract fractions by writing them with a common denominator. They find and describe in words the rule for the next term or nth term of a sequence where the rule is linear. They formulate and solve linear equations with whole-number coefficients. They represent mappings expressed algebraically, and use Cartesian coordinates for graphical representation interpreting general features.

Level 7 (Year 9, high ability, grade 8, age 14)

When making estimates, pupils round to one significant figure and multiply and divide mentally. They understand the effects of multiplying and dividing by numbers between 0 and 1. They solve numerical problems involving multiplication and division with numbers of any size, using a calculator efficiently and appropriately. They understand and use proportional changes, calculating the result of any proportional change using only multiplicative methods. They find and describe in symbols the next term or nth term of a sequence where the rule is quadratic. They use algebraic and graphical methods to solve simultaneous linear equations in two variables.

Programmes of Study
Handling data: Year 6

- Use the language associated with probability to discuss events, including those with equally likely outcomes.
- Solve a problem by representing, extracting and interpreting data in tables, graphs, charts and diagrams, including those generated by a computer, for example:
  – line graphs (e.g. for distance–time, for a multiplication table, a conversion graph, a graph of pairs of numbers adding to 8);
  – frequency tables and bar charts with grouped discrete data (e.g. test marks 0–5, 6–10, 11–15…).
- Find the mode and range of a set of data.
  Begin to find the median and mean of a set of data.

Probability: Year 7

- Use vocabulary and ideas of probability, drawing on experience.
- Understand and use the probability scale from 0 to 1; find and justify probabilities based on equally likely outcomes in simple contexts; identify all the possible mutually exclusive outcomes of a singe event.
- Collect data from a simple experiment and record in a frequency table; estimate probabilities based on this data.
- Compare experimental and theoretical probabilities in simple contexts.

Probability: Year 8

- Use the vocabulary of probability when interpreting the results of an experiment; appreciate that random processes are unpredictable.
- Know that if the probability of an event occurring is p, then the probability of it not occurring is 1 – p ; find and record all possible mutually exclusive outcomes for single events and two successive events in a systematic way, using diagrams and tables.
- Estimate probabilities from experimental data; understand that:
  – if an experiment is repeated there may be, and usually will be, different outcomes;
  – increasing the number of times an experiment is repeated generally leads to better estimates of probability.
- Compare experimental and theoretical probabilities in different contexts.

Probability: Year 9

- Use the vocabulary of probability in interpreting results involving uncertainty and prediction.
- Identify all the mutually exclusive outcomes of an experiment; know that the sum of probabilities of all mutually exclusive outcomes is

1 and use this when solving problems.

- Estimate probabilities from experimental data; understand relative frequency as an estimate of probability and use this to compare outcomes of experiments.
- Compare experimental and theoretical probabilities in a range of contexts; appreciate the difference between mathematical explanation and experimental evidence.

Probability: Year 9 Exceptional Performance

- Understand relative frequency as an estimate of probability and use this to compare outcomes of experiments.

Appendix 2: Probability Tests for 14 Year Olds 2001. Comparisons with the Tests from 1996 to 1999 QCA 2007

1 The performance of pupils at level 4 on *Tokens*, where they considered probabilities relating to drawing gold or silver tokens from a bag, was good and showed a sound understanding of this context and probabilities expressed in words. Pupils who achieved level 5 and above could confidently find the probability of an event not happening, given the probability of its happening, though this concept is at level 6 of the national curriculum. Another question, *Cereal,* was targeted at level 5 and concerned writing probabilities relating to four equally likely events. This question was found straightforward for pupils at level 5. It was pleasing that few pupils lost marks through incorrect notation for probability.

2 In part (a) of the question *Spinners*, pupils were shown two spinners. The first, a regular pentagon, was divided into five similar triangles numbered 1 to 5. The second, a regular hexagon, was divided into six similar triangles numbered 1 to 6. In part (b) pupils were shown two similar hexagonal spinners although one was larger than the other. At a lower level, pupils' explanations to parts (a) and (b) revealed some common misconceptions among pupils at level 4. In part (a), a common incorrect response was an indication that it did not matter which spinner you used to have the best chance of getting 1. This suggests, perhaps, that these pupils had seen the sections labelled 1 as having the same angle at the centre. A common incorrect reason for choosing the correct spinner A was that the arrow on A was closer to 1. In part (b), the common error was to indicate the larger spinner. This error was made by almost 40% of pupils at level 4 and over 20% of those at level 6 and it suggests that pupils have compared the areas of the sections labelled 1 on both spinners, rather than the angles at the centres of the sections.

3 *Coloured Cubes* was also targeted at level 6 and in part (c) given an unspecified number of cubes that are either black or red and the probability that one taken at random being red is 1/5, pupils at level 6 found some difficulty understanding that the same probability could arise from different total numbers of cubes. In part (d) pupils were given the context of 20 cubes with at least one of each of three different colours. Given the probability of drawing a cube of one colour at random they were asked to find the greatest possible number of a second colour. Of pupils awarded level 7, only just over half were awarded both marks for part (d).

4 At level 8, *Pots* assessed the probability of combined events. Pupils were given the probability 0.03 of a pot cracking when fired. In part (a) they were asked to calculate the probability of both pots cracking when two were fired. The common incorrect response in part (a) was that of 0.06, obtained by doubling 0.03 rather than squaring it. This error was made by almost 40% of pupils at level 7. However, of those pupils at level 8, over 65% gained the mark; and a further 20% had shown the calculation 0.03×0.03 in their working although their final answers were incorrect. Part (b) asked for the probability of only one pot cracking when two were fired. This was found difficult and fewer than 10% of pupils gaining level 8 overall were awarded both marks. However, a further 40% gained one of the two marks for an answer of 0.029(1), which indicates that pupils had forgotten to take order into account. The common incorrect response from pupils at level 7 was 0.03, based on the calculation 0.03×1.